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13. ABSTRACT

This report stems from the observation of data whose statistical parameters, such as the means and variances, mask essential information like transients or possibly target signal levels imbedded in background noise. Transient information in this report implies that a large displacement of a very small percentage of samples exists in the data. The report shows that this information contained within the means and variances of ocean ambient noise data can be removed by the manipulation of the differences of some of the higher order moments. Techniques in this analysis were specifically oriented to help solve detection problems currently faced by the ASW community.

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DEPARTMENT OF THE NAVY
NAVAL AIR DEVELOPMENT CENTER
JOHNSVILLE
WARMINSTER, PA. 18974

Systems Analysis and Engineering Department

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# Ocean Ambient Noise Statistics for the Detection of Transients

FINAL REPORT

ASW-141-71-1

Prepared by: B. Polin
B. Polin

Reviewed by: T. E. Willey
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W. H. Raber
Director

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## INTRODUCTION

This report presents a mathematical procedure designed to identify the difference between two similar sets of data used in statistical analysis. It was developed specifically to help solve detection problems currently faced by the ASW (Anti-Submarine Warfare) community. The procedure applies to data whose statistical parameters, such as the means and variances, mask essential information like transients or possibly target signal levels imbedded in background noise. Transient information in this report implies that there is a large displacement of a very small percentage of the data.

In this procedure, the variability in the data is isolated and defined by making a comparison of the statistics of a density function from one time interval to the next. This is accomplished primarily by the manipulation of the differences between several higher order moments. The analysis presented herein indicates that the important factors in this comparison are the number of data observations that have been shifted in time relative to a reference density function and the effective amplitude. The effective amplitude will be defined as a displacement that will replace the effect of the shift of several observations that are associated with the transient. This type of computation can be referred to as a mathematical "glimpse" of an approximation of the variation in a density function over time.

## SUMMARY OF RESULTS

The report considers the problem of extracting transient information, that is, the effective amplitude, from large quantities of ambient noise data without examining each data sample. The effective amplitude is generally not extractable from statistical parameters such as the mean and the variance.

The task of finding the effective amplitude is accomplished by identifying the important factors that describe the difference between two similar density functions derived from data with overlapping time intervals. This difference is dependent on the effective amplitude and the associated data samples within this transient. The effective amplitude is then evaluated by the manipulation of the differences found among higher order moments. The knowledge of its value enables the computation of the number of data samples associated with the transient. Therefore, it is shown that knowledge of the factors describing this difference will enable the removal of transient information from large quantities of data without the necessity of examining each data sample.

## CONCLUSIONS

The major conclusions drawn from this analysis are stated here:

1. This development will enable the removal of transient information from large quantities of data without the necessity of examining each data sample.

2. This mathematical development is applicable to ocean ambient noise data since these statistics display temporal fluctuation. The properties of the technique that apply are:

a. The requirement of highly calibrated measurement equipment is greatly reduced since the analysis employs differences which exist among means, variances, and higher order moments.

b. The information derived from the analysis consists of the effective amplitude and the associated number of data samples that change over intervals of time.

c. Application of the mathematical techniques presented herein is directly applicable to the changing pattern of differences among higher order moments of an ambient noise plus signal distribution in a field of sonobuoys resulting from the effects of a traversing target. Therefore, this procedure affords a conceptual approach to the detection of submarines.

3. A by-product of this analysis yields the interrelations of the moments between two similar histograms.

4. The knowledge of the effective amplitude and its associated number of data samples can be used as a measure for determining the temporal fluctuation between two similar histograms.

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# DISCUSSION

## I. Background

Ocean ambient noise is one of the least understood of the basic parameters required to effectively model acoustic detection. One of the difficulties in describing this phenomenon is that the statistical characteristics are from a nonstationary process, that is, the statistical properties are time dependent, as observed in references (a) and (b). This effect can be attributed to at least two factors - shipping in the ocean environment, and environmental changes in the ocean. Therefore, the means and variances of the resulting distribution from these data are not always adequate for describing central tendencies and variabilities.

Additional information contained within the data can be unmasked by considering the higher order normalized moments. For example, information about the shape of the distribution can be obtained from the knowledge of the third and fourth moments. Also, as observed in reference (b), when a transient condition exists in the data, the higher order moments become very large.

The purpose of this report is to introduce the development of a mathematical technique that may be applied to removing some of the difficulties in describing ambient noise. More important is the potential application of this mathematical development in extracting transient information from large quantities of ambient noise data without examining each data sample.

## II. Methodology

This analysis is based on the premise that when the comparison is made between two similar density functions with overlapping time intervals, the resulting differences of the means, variances, and higher order moments will reflect this change. These differences are more noticeable as the moments become higher. The procedure developed will decompose the change into the effective amplitude and the associated number of samples. Thus, the analysis determines whether the change or shift in the density function is due to a transient condition or a small amplitude with a large number of samples, or some combination conditions between these two extremes.

The basic equations used in this development will be the sample mean, $\mu$

$$\mu = \frac{\sum_{i}^{N} X_i}{N}$$

i is a running index from 1 to N
and N is the number of data samples

$X_i$ is the variate in the data.

and the equation for the $n^{th}$ moment about the mean is

$$\mu_n = \frac{\sum_{i}^{N} (X_i-\mu)^n}{N}$$

where n is the order of the moment

If n is equal to 2, then $\mu_2$ is the variance defined by the symbol $\sigma^2$.

## III. Analysis Development

### A. Introduction

This analysis shows in detail how the mean and variance are altered by the shift of one of the variates of a given data set. This is followed by a geometric interpretation of the shifted variance equation which is then modified to include the condition of having more than one shifted data sample. The final equations describing the shift of the 3rd and 4th moments will be presented, leaving the details for the appendix. Manipulation of the shifted variance equation and of the 3rd and 4th moment equations will lead to the solution for expressions of the effective amplitude and the associated number of data points that were shifted.

### B. Shifted Mean

The analysis begins by defining for an arbitrary distribution a reference mean, $\mu_r$

$$\mu_r = \frac{\sum_{i}^{N} X_i}{N} = \frac{X_1 + X_2 + \ldots + X_{N-j+1} + \ldots + X_N}{N} \qquad (1)$$

In a short interval of time one of these observations, $X_{N-j+1}$ shifts $A_{N-j+1}$ units while the other samples maintain their original values. This new observation is equal to $\mathcal{X}_{N-j+1} = X_{N-j+1} + A_{N-j+1}$.

The statistics of the arbitrary distribution become altered due to the variate $\mathcal{A}_{N-j+1}$. A new mean, $\mu_a$, of the altered arbitrary distribution equals

$$\mu_a = \frac{\sum_i^N X_i}{N} = \frac{X_1 + X_2 + \ldots + \mathcal{A}_{N-j+1} + \ldots + X_N}{N} \tag{2}$$

$$= \frac{X_1 + X_2 + \ldots + X_{N-j+1} + A_{N-j+1} \ldots + X_N}{N}$$

$$= \frac{X_1 + X_2 + \ldots + X_{N-j+1} + \ldots + X_N}{N} + \frac{A_{N-j+1}}{N}$$

$$\mu_a = \mu_r + \mu_A;\ \text{where}\ \mu_A = \frac{A_{N-j+1}}{N} \tag{3}$$

Thus, the new mean of the altered distribution equals the reference mean $\mu_r$, plus the shift of the mean, $\mu_A$.

C. Shifted Variance

The reference variance, $\sigma_r^2$ equals

$$\sigma_r^2 = \mu_{2r} = \frac{\sum_i^N (X_i - \mu_r)^2}{N} \tag{4}$$

The new variance, $\sigma_a^2$, of the altered arbitrary distribution due to the shifted variate, $\mathcal{A}_{N-j+1}$, equals

$$\sigma_a^2 = \frac{1}{N} \sum_i^N (X_i - \mu_a)^2 \tag{5}$$

$$= \frac{1}{N} [(X_1 - \mu_r - \mu_A)^2 + (X_2 - \mu_r - \mu_A)^2 + \ldots + (X_{N-j+1} + A_{N-j+1} - \mu_r - \mu_A)^2 + \ldots + (X_N - \mu_r - \mu_A)^2]$$

$$= \frac{1}{N} [X_1^{\ 2} - 2\mu_r X_1 + \mu_r^{\ 2} - 2\mu_A X_1 + 2\mu_r \mu_A + \mu_A^{\ 2}$$

$$+ X_2^{\ 2} - 2\mu_r X_2 + \mu_r^{\ 2} - 2\mu_A X_2 + 2\mu_r \mu_A + \mu_A^{\ 2}$$

$$\vdots$$

$$+ X^2_{N-j+1} - 2\mu_r X_{N-j+1} + \mu_r^{\ 2} - 2\mu_A X_{N-j+1} + 2\mu_r \mu_A + \mu_A^{\ 2}$$

$$+ 2(X_{N-j+1} - \mu_r - \mu_A) A_{N-j+1} + A^2_{N-j+1}$$

$$\vdots$$

$$+ X_N^{\ 2} - 2\mu_r X_N + \mu_r^{\ 2} - 2\mu_A X_N + 2\mu_r \mu_A + \mu_A^{\ 2}]$$

$$\sigma_a^{\ 2} = \frac{\sum_i^N (X_i^{\ 2} - 2\mu_r X_i + \mu_r^{\ 2})}{N} - 2\mu_A \frac{\sum_i^N X_i}{N} + \frac{2\mu_A \mu_r N}{N} + \frac{N \mu_A^{\ 2}}{N}$$

$$+ \frac{[2(X_{N-j+1} - \mu_r - \mu_A) A_{N-j+1} + A^2_{N-j+1}]}{N}$$

The first term of this expression for $\sigma_a^2$ is the reference variance $\sigma_r^2$ since

$$\sigma_r^{\ 2} = \frac{1}{N} \Sigma (X_i^{\ 2} - 2\mu_r X_i + \mu_r^{\ 2}) = \frac{\sum_1^N (X_i - \mu_r)^2}{N}$$

The second two terms equal zero since

$$-2\mu_A \frac{\sum_1^N X_i}{N} + \frac{2\mu_r \mu_A N}{N} = -2\mu_A \mu_r + 2 \mu_r \mu_A = 0$$

The third and final term becomes the difference between two squares, since

$$\frac{2(X_{N-j+1} - \mu_r - \mu_A)A_{N-j+1} + A^2_{N-j+1}}{N} = \frac{2(X_{N-j+1} - \mu_a)A_{N-j+1} + A^2_{N-j+1}}{N} \quad (6)$$

$$= \frac{(X_{N-j+1} - \mu_a + A_{N-j+1})^2 - (X_{N-j+1} - \mu_a)^2}{N}$$

Thus, the final equation for the shifted variance is:

$$\sigma_a^2 = \sigma_r^2 + \mu_A^2 + \frac{(X_{N-j+1} - \mu_a + A_{N-j+1})^2 - (X_{N-j+1} - \mu_a)^2}{N} \quad (7)$$

The variance for the altered arbitrary distribution equals the reference variance plus the shift of the mean and the term that is the difference between two squares. This latter term will be referred to as the second moment arm. This term will be defined by the symbol, $Z_N^2$. It equals:

$$Z_N^2 = \frac{(X_{N-j+1} - \mu_a + A_{N-j+1})^2 - (X_{N-j+1} - \mu_a)^2}{N}$$

As N increases, $\mu_A$ decreases since $\mu_A = \frac{A_{N-j+1}}{N}$ and the second moment arm also decreases. Therefore, as N becomes very large, $\sigma_a^2$ approaches $\sigma_r^2$ which is to be expected.

D. Implication of the Shifted Variance

Before examining the treatment of more than one variate being shifted, it is worthwhile to consider the geometric implication for the second moment arm

$$Z_N^2 = \frac{(X_{N-j+1} - \mu_a + A_{N-j+1})^2 - (X_{N-j+1} - \mu_a)^2}{N}$$

In this discussion, let the data sample $X_{N-j+1}$ be located at the edge of the distribution and be shifted a distance $A_{N-j+1}$ units as indicated in figure 1.

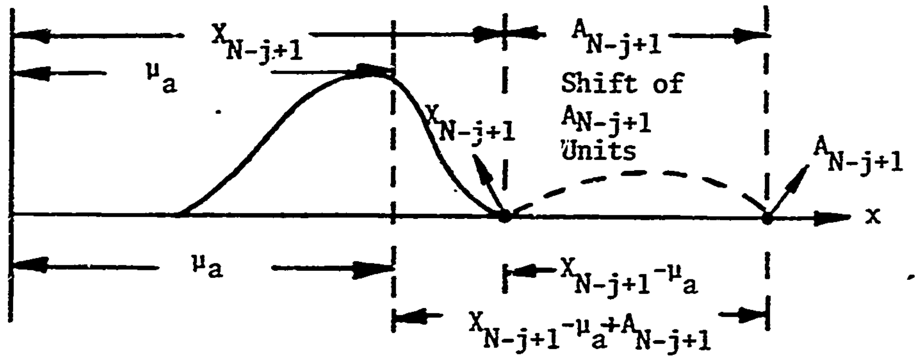


FIGURE 1. GEOMETRIC INTERPRETATION OF THE SHIFTED VARIANCE

The first term of $Z_N^2$ is equal to the square of the distance of the shifted variate of $X_{N-j+1}$ from the edge of the density function out to a distance $A_{N-j+1}$ units, minus the value of the altered mean $\mu_a$; that is, a distance of $(X_{N-j+1}-\mu_a + A_{N-j+1})$ units. The second term is equal to the square of the distance from the edge of the distribution $X_{N-j+1}$, minus the distance of the altered mean $\mu_a$; that is, a distance of $(X_{N-j+1} - \mu_a)$ units. As observed, both of these distances are independent of the origin.

The moment arm associated with the shift of the variate for the higher order moments becomes:

$$Z_N^n = \frac{(X_{N-j+1} - \mu_a + A_{N-j+1})^n - (X_{N-j+1} - \mu_a)^n}{N}$$

The details of the development for this expression are left for appendix A. This particular term $Z_N^n$ is dominant in the computation of the higher order moments. As the order of the moments increases, the shift of the distance $A_{N-j+1}$ units from the edge of the density function causes the moment arm to increase also.

Another term that is important for this analysis will be referred to as the second moment constant. It is based on the shifted variance equation, $\sigma_a^2$.

$$\sigma_a^2 = \sigma_r^2 + \mu_A^2 + \frac{(X_{N-j+1} - \mu_a + A_{N-j+1})^2 - (X_{N-j+1} - \mu_a)^2}{N} \tag{7}$$

or

$$\sigma_a^2 - \sigma_r^2 - \mu_A^2 = \frac{\cancel{(X_{N-j+1} - \mu_a)^2} + 2A_{N-j+1}(X_{N-j+1} - \mu_a) + A_{N-j+1}^2 - \cancel{(X_{N-j+1} - \mu_a)^2}}{N} \tag{8}$$

Remembering that the shift of the mean is

$$\mu_A = \frac{A_{N-j+1}}{N} = \mu_a - \mu_r$$

The second moment constant, $M_2$, will be defined to equal

$$M_2 = \frac{\sigma_a^2 - \sigma_r^2 - \mu_A^2}{2\,\mu_A} = X_{N-j+1} - \mu_a + \frac{A_{N-j+1}}{2} \tag{9}$$

Again, if $X_{N-j+1}$ is at the edge of a density function and shifted out to a distance of $A_{N-j+1}$ units, then the second moment constant $M_2$ will be the distance from the altered mean out to the distance halfway between the edge of the distribution and the transient. If the shift of the distance is inside the density function, then $A_{N-j+1}$ will be negative. Mathematically, this second moment constant, $M_2$, represents a distance describing a shift of a variate that can be inside or outside the density function and it equals the differences of computed statistics.

E. Treatment of More Than One Data Observation

This section discusses what happens to the altered mean expression, $\mu_a$, and the shifted variance equation, $\sigma_a^2$, when more than one observation has been displaced. The general expression for the altered mean equation will be developed. However, the analysis considers three observations being displaced in the discussion of the shifted variance equation. It should be understood that this development can apply in general to many observations.

As shown by equation (3), the altered mean, $\mu_a$, equals

$$\mu_a = \mu_r + \mu_A \tag{3}$$

or

$$\mu_a = \mu_r + \frac{A_{N-j+1}}{N}$$

where

$$\mu_A = \frac{A_{N-j+1}}{N}$$

However, if there were many observations being displaced, then

$$\mu_A = \frac{A_{N-j+1} + A_{N-j+2} + \dots + A_{N-j+Ne}}{N}$$

$$= \frac{\sum_{i=N-j+m}^{Ne} A_i}{N}$$

where m is a running index from 1 to $N_e$ and $N_e$ is the number of data points being displaced

Let $A_e$ be defined as an amplitude that will replace the effect of the displacement of many observations. It equals

$$A_e = \frac{1}{N_e} \sum_{i=N-j+m}^{Ne} A_i \quad .$$

Now, $\mu_A$ will equal

$$\mu_A = \frac{N_e A_e}{N} \quad . \qquad (10)$$

and $A_e$ will be referred to as an effective amplitude. It will be determined by the use of the differences of the higher order moments. The value of $N_e$ is determined by the knowledge of $\mu_A$ which is equal to the difference between the altered mean and the referenced mean. Therefore, $N_e$ is defined as the effective number of data points that are associated with the shift of the mean $\mu_A$, or the number of data points that significantly contribute to the difference between moments of similar statistics.

The modification of the shifted variance equation for the displacement of three observations will be developed by first referring to equation (5). It should be observed that the reference variance, $\sigma_r^2$, and the shift of the mean, $\mu_A$, will not be modified when considering the shift of more than one data point. The moment arm expression is the term that will have to be modified and the form used (page 5, second term of equation (6)) in this discussion is:

$$Z_N{}^2 = \frac{2A_{N-j+1}}{N}(X_{N-j+1} - \mu_a + \frac{A_{N-j+1}}{2})$$

As discussed earlier, the expression in the bracket, $X_{N-j+1} - \mu_a + \frac{A_{N-j+1}}{2}$ geometrically represents a measured distance halfway between the variate before being shifted and to the distance of the transient. If there were several observations shifted approximately A units, then with each observation, there would be an associated distance shift of the second moment arm. Since considerations are now given to data with more than one variate being shifted, it is worthwhile to consider another term to be defined as $X_e$, the effective distance. It is equal to the average value before the shifting of all variates that are shifted. The techniques to be developed are the approximate value of the effective amplitude, $A_e$, that is related to the shift, and the associated number of data points, $N_e$, associated with the shift.

The modification of the second moment arm, $Z_N{}^2$, due to the shift of few variates that were caused by amplitudes, $A_{N-j+1}$, $A_{N-j+2}$, and $A_{N-j+3}$, is shown below:

$$Z_N{}^2 = \frac{2A_{N-j+1}}{N}[X_{N-j+1} - \mu_a + \frac{A_{N-j+1}}{2}] \tag{12}$$

$$+ \frac{2A_{N-j+2}}{N}[X_{N-j+2} - \mu_a + \frac{A_{N-j+2}}{2}] + \frac{2A_{N-j+3}}{N}[X_{N-j+3} - \mu_a + \frac{A_{N-j+3}}{2}]$$

The terms in the brackets of the expression for $Z_N{}^2$ are also associated with distances that are shown below in figure 2.

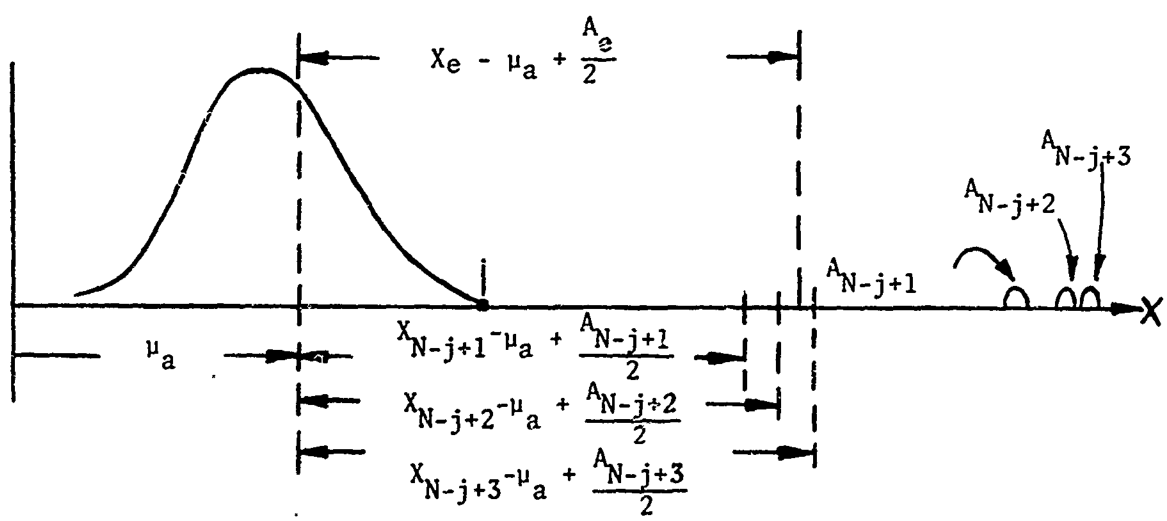


FIGURE 2. GEOMETRIC INTERPRETATION OF THE SHIFTED VARIATE FOR MORE THAN ONE VARIATE

The technique developed in this section shows how to solve for an average or an effective amplitude, $A_e$. Once $A_e$ is known, $N_e$ can be computed by the knowledge of $\mu_A$. The additional term, $X_e$, can also be evaluated. This distance $X_e$ will normally be within the density function. Figure 2 also shows the relative location of $X_e$ as well as $A_e/2$.

Since this computation considers an effective amplitude, $A_e$, and an effective distance, $X_e$, errors resulting from the approximation exist in the analysis. However, in the case of a transient, where the shifts are large, the approximation error should be very small.

In order to solve for the terms $X_e$ and $A_e$, they have to be introduced within the brackets of equation (12). This can be accomplished by setting each bracket term plus a differential equal to $(X_e-\mu_a+A_e/2)$ as shown below.

$$[X_e - \mu_a + \frac{A_e}{2}] = [X_{N-j+1} - \mu_a + \frac{A_{N-j+1}}{2}] + \delta_{N-j+1} \tag{13}$$

$$= [X_{N-j+2} - \mu_a + \frac{A_{N-j+2}}{2}] + \delta_{N-j+2}$$

$$= [X_{N-j+3} - \mu_a + \frac{A_{N-j+3}}{2}] + \delta_{N-j+3}$$

The differential term will adjust the X's and the A's to be equal to $X_e$ and $A_e$, respectively. Thus, introducing an error into the bracket terms, and rewriting equation (12), we obtain:

$$Z_N^{\ 2} = \frac{2A_{N-j+1}}{N} [X_e - \mu_a + \frac{A_e}{2} - \delta_{N-j+1}] + \frac{2A_{N-j+2}}{N} [X_e - \mu_a + \frac{A_e}{2} - \delta_{N-j+2}] \tag{14}$$

$$+ \frac{2A_{N-j+3}}{N} [X_e - \mu_a + \frac{A_e}{2} - \delta_{N-j+3}]$$

$$= \frac{2(A_{N-j+1} + A_{N-j+2} + A_{N-j+3})}{N} [X_e - \mu_a + \frac{A_e}{2}]$$

$$- \frac{2}{N} (A_{N-j+1}\, \delta_{N-j+1} + A_{N-j+2}\, \delta_{N-j+2} + A_{N-j+3}\, \delta_{N-j+3})$$

Since

$$\mu_A = \frac{N_e A_e}{N} = \frac{A_{N-j+1} + A_{N-j+2} + A_{N-j+3}}{N}$$

equation (12) becomes:

$$Z_N^2 = \frac{2N_e A_e}{N} \left[ (X_e - \mu_a + \frac{A_e}{2}) - \frac{A_{N-j+1}\,\delta_{N-j+1} + A_{N-j+2}\,\delta_{N-j+2} + A_{N-j+3}\,\delta_{N-j+3}}{A_{N-j+1} + A_{N-j+2} + A_{N-j+3}} \right] \quad (15)$$

where the $\delta$'s can be both positive and negative values. The errors for the second moment arm should be very small and therefore are ignored.

The second moment constant in the case of several shifts would be equal to:

$$M_2 = \frac{\sigma_a^2 - \sigma_r^2 - \mu_A^2}{2\mu_A} = X_e - \mu_a + \frac{A_e}{2} \quad (16)$$

Thus, there is one equation with two unknowns.

In order to solve for $X_e$ and $A_e$, additional equations are needed. They are found by introducing a shift of a variate into the higher order moment equations. The third and fourth higher order shifted moment equations are developed in appendix B and are given below.

The third moment shift equation:

$$\mu_{3a} = \mu_{3r} - 3\mu_A\sigma_r^2 - \mu_A^3 + Z_N^3 \quad (17)$$

The fourth moment shift equation:

$$\mu_{4a} = \mu_{4r} - 4\mu_A\,\mu_{3r} + 6\mu_A^2\,\sigma_r^2 + \mu_A^4 + Z_N^4 \quad (18)$$

and

$$Z_N^n = \frac{N_e}{N}\left[(X_e - \mu_a + A_e)^n - (X_e - \mu_a)^n\right]$$

As mentioned above, the approximation errors in the above equations are considered negligible. The subscripts (a) and (r) refer to the altered distribution and the reference distribution, respectively.

The third and fourth moment constants, which are developed in appendix B, are given below. The third moment constant, neglecting the approximation error,

$$M_3 = \frac{\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3}{3\mu_A} = (X_e - \mu_a + \frac{A_e}{2})^2 + \frac{1}{3}(\frac{A_e}{2})^2 \qquad (19)$$

For the fourth moment constant,

$$M_4 = \frac{\mu_{4a} - \mu_{4r} + 4\mu_A \mu_{3r} - 6\mu_A^2 \sigma_r^2 - \mu_A^4}{4\mu_A} \qquad (20)$$

$$= (X_e - \mu_a + \frac{A_e}{2}) [(X_e - \mu_a + \frac{A_e}{2})^2 + (\frac{A_e}{2})^2]$$

Thus, two more moment constants relating $X_e$ and $A_e$ exist. It is observed that since two unknowns exist, there are more equations than needed. Indications are that the higher the order of the moment constants used, the more accurate the answer for $X_e$ and $A_e$.

F. Solutions for the Differences

Solving for $X_e$, $A_e$ and $N_e$ using the constants $M_2$ and $M_3$, the second moment constant becomes

$$M_2 = \frac{\sigma_a^2 - \sigma_r^2 - \mu_A^2}{2\mu_A} = X_e - \mu_a + \frac{A_e}{2} .$$

For the third moment constant:

$$M_3 = \frac{\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3}{3\mu_A} = (X_e - \mu_a + \frac{A_e}{2})^2 + \frac{1}{3}(\frac{A_e}{2})^2$$

$$M_3 = (M_2)^2 + \frac{1}{3}(\frac{A_e}{2})^2$$

Solving for $A_e$

$$A_e = \sqrt{12(M_3 - M_2^2)}$$

Hence,

$$N_e = \frac{N \mu_A}{A_e} \text{ and } X_e = M_2 + \mu_a - \frac{A_e}{2}$$

where $N_e$ is the associated number of samples and $X_e$ = the relative position of the variate before the amplitude of $A_e$ units is added to the data.

Solving for $A_e$ using $M_2$, $M_3$ and $M_4$

$$M_4 = \frac{\mu_{4a} - \mu_{4r} + 4\mu_A\ \mu_{3r} - 6\mu_A^2\ \sigma_r^2 - \mu_A^4}{4\mu_A}$$

$$= (X_e - \mu_a + \frac{A_e}{2})\ [(X_e - \mu_a + \frac{A_e}{2})^2 + (\frac{A_e}{2})^2]$$

However, the bracket term is:

$$[(X_e - \mu_a + \frac{A_e}{2})^2 + (\frac{A_e}{2})^2] = [(X_e - \mu_a + \frac{A_e}{2})^2 + \frac{1}{3}\ (\frac{A_e}{2})^2 + \frac{2}{3}\ (\frac{A_e}{2})^2]$$

$$= M_3 + \frac{2}{3}\ (\frac{A_e}{2})^2$$

The fourth moment constant equals

$$M_4 = (X_e - \mu_a + \frac{A_e}{2})[M_3 + \frac{2}{3}\ (\frac{A_e}{2})^2]$$

$$= M_2\ [M_3 + \frac{2}{3}\ (\frac{A_e}{2})^2]$$

Solving again for $A_e$,

$$A_e = \sqrt{6(M_4/M_2 - M_3)}$$

It is observed that $M_4$, $M_3$, and $M_2$ are functions of the difference of the computed moments and are not dependent on absolute measured values. Both $A_e$ and $N_e$ are dependent on the moment constants and independent of absolute values. They are the common factors that interrelate two similar sets of statistics. Therefore, they identify the difference between two overlapping density functions. These ingredients that describe the differences can be an aid in the analysis of time varying statistical data such as ocean ambient noise.

## IV. Conclusions of the Analysis

Techniques developed from this analysis yield a new tool that extracts the differences between two sets of similar statistics. The extraction will be the displacement ($A_e$) that will replace the effect of a shift of several observations and the number of observations ($N_e$). These two quantities reflect the changes that alter statistics that are measured from one time interval to the next. The analysis models the occurrence of these effects even if they are deeply imbedded in the temporal ocean ambient noise data. The important feature of this tool is that the gross changes in the data can be removed without the necessity of examining every data sample.

Consider the situation of ocean ambient noise levels being monitored by the field of sonobuoys. If a target traverses this field, the time varying averages of the ambient noise levels may mask information relating to its presence. However, the changing means, variances, and some of the differences of the higher order moments will reflect the number of data samples, $N_e$, that vary in time throughout the field. Possible target presence can be indicated by an increasing value of $N_e$ relative to the data of the other buoys. If a transient were present, the value of $A_e$ would increase. These measurements are dependent on the relative changes of moments and not the absolute value of highly calibrated buoys at each position in the field. Therefore, the use of this mathematical technique affords a conceptual approach for the detection of submarine targets.

The tool can be used in aiding the analysis of temporal fluctuations associated with ocean ambient noise data. A by-product of this analysis yields interrelations of moments between similar density functions. Therefore, the tool can aid in measuring the departures of time varying statistics.

## REFERENCES

(a) Arase, T. and Arase, E. M., "Deep Sea Ambient Noise Statistics," The Journal of the Acoustical Society of America, 15 Apr 1968

(b) Naval Air Development Center Report No. NADC-SD-7905 "A Statistical Investigation of Low Frequency Ocean Ambient Noise" (U) of 4 Dec 1970, CONFIDENTIAL

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# APPENDIX A

## THE SHIFTED HIGHER ORDER MOMENTS

In the development of equation (7), the shifted variance equation is expressed as

$$\sigma_a^{\,2} = \sigma_r^{\,2} + \mu_A^{\,2} + Z_N^{\,2} \tag{7}$$

It is observed that $\sigma_a^2$ is a function of the reference variance, a term due to the shift of the mean and the moment arm. It will also be shown that the higher order moment equations will have the same form.

In order to simplify the development of the higher order moment equations, the Kronecker delta will be introduced into this section of the paper. This term has the following properties.

$$\delta_{i,N-j} = \begin{cases} 1 & i = N-j \\ 0 & i \neq N-j \end{cases}$$

Now, let the variate $X_i$ be equal to

$$X_i = X_i + \delta_{i,N-j} \cdot A_{N-j}$$

Hence, when the summation of $X_i$ from i to N is performed, the amplitude of A units that causes the shift in variate will only have a value when $i = N-j$.

### Development of the Shifted Third Moment Equation

Remembering that $\mu_a = \mu_r + \mu_A$ and starting with the third moment equation

$$\mu_{3a} = \frac{1}{N}\sum_i^N (X_i-\mu_a)^3 = \frac{1}{N}\sum_i^N (X_i+ \delta_{i,N-j}\, A_{N-j} - \mu_r-\mu_A)^3 \tag{A-1}$$

Let $\qquad z_i = X_i-\mu_r-\mu_A$ and $y_i = X_i-\mu_r$

Now consider the third moment expression when i=N-j, then

$$\mu_{3a}\Big|_{i=N-j} = \frac{1}{N}(z_{N-j} + A_{N-j})^3$$

$$= \frac{1}{N}\left[z_{N-j}^3 + 3A_{N-j}\, z_{N-j}^2 + 3A_{N-j}^2\, z_{N-j}^1 + A_{N-j}^3\right]$$

$$= \frac{1}{N}\left[(X_{N-j}-\mu_r-\mu_A)^3 + 3A_{N-j}(X_{N-j}-\mu_r-\mu_A)^2 + 3A_{N-j}^2(X_{N-j}-\mu_r-\mu_A)^1 + A_{N-j}^3\right]$$

The term associated with the moment arm will be

$$Z_N{}^3 = \frac{1}{N}\left[3A_{N-j}(X_{N-j}-\mu_a)^2 + 3A_{N-j}^2(X_{N-j}-\mu_a) + A_{N-j}^3\right] \qquad \text{(A-2)}$$

Adding and subtracting yields $(X_{N-j} - \mu_a)^3$

$$Z_N{}^3 = \frac{1}{N}\left[(X_{N-j}-\mu_a)^3 + 3A_{N-j}(X_{N-j}-\mu_a)^2 + 3A_{N-j}^2(X_{N-j}-\mu_a) \right. \qquad \text{(A-3)}$$

$$\left. + A_{N-j}^3 - (X_{N-j} - \mu_a)^3\right]$$

$$= \frac{(X_{N-j}-\mu_a + A_{N-j})^3 - (X_{N-j} - \mu_a)^3}{N}$$

Thus, the third moment expression when $i = N-j$ is

$$\mu_{3a}\Big|_{i=N-j} = \frac{1}{N}(X_i-\mu_r-\mu_A)^3 + Z_N{}^3$$

$$= \frac{1}{N}(y_i-\mu_A)^3 + Z_N{}^3$$

Now, consider the expression of $\mu_{3a}$ with the amplitude $A_{N-j}$ term being removed from the summation

$$\mu_{3a} = \frac{1}{N}\sum_i^N (yi-\mu_A)^3 + Z_N{}^3$$

$$= \frac{1}{N}\sum_i^N y_i{}^3 - \frac{3\mu_A}{N}\sum_i^N y_i{}^2 + \frac{3\mu_A}{N}\sum_i^N y_i - \frac{N\mu_A{}^3}{N} + Z_N{}^3$$

$$= \mu_{3r} - 3\mu_A\,\sigma_r{}^2 - \mu_A{}^3 + Z_N{}^3$$

The summation of

$$\frac{1}{N}\sum_i^N y_i = \frac{1}{N}\sum_i^N (X_i-\mu_r) = 0$$

Therefore, the third moment equation due to a shift in the variate by A units is

$$\mu_{3a} = \mu_{3r} - 3\mu_A \sigma_r^{\,2} - \mu_A^{\,3} + Z_N^{\,3} \tag{A-4}$$

Considering the case when more than one variate has been shifted but ignoring the approximation errors, $Z_N^3$ becomes

$$Z_N^{\,3} = \frac{N_e}{N} [(X_e - \mu_a + A_e)^3 - (X_e - \mu_a)^3] \tag{A-5}$$

Development of the Shifted Fourth Moment Equation

Starting with the moment equation

$$\mu_{4a} = \frac{1}{N} \sum_i^N (X_i - \mu_a)^4 = \frac{1}{N} \sum_i^N (X_i - \mu_r - \mu_A + \delta_{N-j}\, A_{N-j})^4$$

but first by just considering the term, $\mu_{4a}$, when $i = N-j$, we obtain

$$\mu_{4a}\Big|_{i=N-j} = \frac{1}{N} \sum_i^N (z_{N-j} + A_{N-j})^4$$

$$= \frac{1}{N} (z^4_{N-j} + 4A_{N-j}\, z^3_{N-j} + 6A^2_{N-j}\, z^2_{N-j} + 4A^3_{N-j}\, z_{N-j} + A^4_{N-j})$$

The moment arm part of the expression is

$$Z_N^{\,4} = \frac{1}{N} (4A_{N-j}\, z^3_{N-j} + A_{N-j}\, z^2_{N-j} + 4A_{N-j}\, z^3_{N-j} + A^4_{N-j})$$

$$= \frac{1}{N} (z^4_{N-j} + 4A\, z^3_{N-j} + \sigma A^2\, z^2_{N-j} + 4A^3\, z_{N-j} + A^4 - z^4_{N-j})$$

$$= \frac{(z_{N-j} + A_{N-j})^4 - z^4_{N-j}}{N}$$

$$Z_N^{\,4} = \frac{(X_{N-j} - \mu_a + A_{N-j})^4 - (X_{N-j} - \mu_a)^4}{N};\ z_{N-j} = X_{N-j} - \mu_a \tag{A-6}$$

Considering the case when more than one variate has been shifted but ignoring the approximation errors for $Z_N{}^4$

$$Z_N{}^4 = \frac{N_e}{N} [(X_e - \mu_a + A_e)^4 - (X_e - \mu_a)^4] \tag{A-7}$$

Thus, the fourth moment expression when i - N-j becomes

$$\mu_{4a}\Big|_{i-N-j} = \frac{1}{N} (X_i - \mu_r - \mu_A)^4 + Z_N{}^4$$

$$= \frac{1}{N} (y_i - \mu_A)^4 + Z_N{}^4$$

Now, consider the expansion of $\mu_{4a}$ with the amplitude $A_{N-j}$ not being considered in the summation

$$\mu_{4a} = \frac{1}{N} \sum_i^N (y_i - \mu_A)^4 + Z_N{}^4$$

$$= \frac{1}{N} \sum_i^N y_i{}^4 - \frac{4\mu_A}{N} \sum_i^N y_i{}^3 + \frac{6\mu_A{}^2}{N} \sum_i^N y_i{}^2 - \frac{4\mu_A{}^3}{N} \sum_i^N y_i{}^1 + \frac{N\mu_A{}^4}{N} + Z_N{}^4$$

$$\mu_{4a} = \mu_{4r} - 4\mu_A \mu_{3r} + 6\mu_A{}^2 \sigma_r{}^2 + \mu_A{}^4 + Z_N{}^4 \tag{A-8}$$

The term

$$\sum_i^N y_i = \frac{1}{N} \sum_i^N (X_i - \mu_r) = 0$$

As can be seen in this development, the expression for the $n^{th}$ order moment of the shifted variate is

$$\mu_{na} = \frac{1}{N} \sum_i^N (y_i - \mu_A)^n + Z_N{}^n; \quad \text{where } y_i = X_i - \mu_r \tag{A-9}$$

# APPENDIX B

## THE MOMENT CONSTANTS

The third moment constant will be developed by rewriting equation (A-4) and ignoring the approximation

$$\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3 = \frac{N_e}{N} [(X_e - \mu_a + A_e)^3 - (X_e - \mu_a)^3] \quad \text{(B-1)}$$

$$\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3 = \frac{N_e}{N} [3A_e(X_e - \mu_a)^2 + 3A_e^2(X_e - \mu_a) + A_e^3]$$

$$= \frac{3N_e A_e}{N} [(X_e - \mu_a)^2 + A_e(X_e - \mu_a) + \frac{A_e^2}{3}]$$

$$= \frac{3N_e A_e}{N} [(X_e - \mu_a + \frac{A_e}{2})^2 - \frac{A_e^2}{4} + \frac{A_e^2}{3}]$$

$$\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3 = \frac{3N_e A_e}{N} [(X_e - \mu_a + \frac{A_e}{2})^2 + \frac{1}{3} (\frac{A_e}{2})^2]$$

The third moment constant becomes

$$M_3 = \frac{\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3}{3 N_e A_e / N} = (X_e - \mu_a + \frac{A_e}{2})^2 + \frac{1}{3} (\frac{A_e}{2})^2$$

But $\mu_A = \frac{N_e A_e}{N}$ and

$$M_3 = \frac{\mu_{3a} - \mu_{3r} + 3\mu_A \sigma_r^2 + \mu_A^3}{3\mu_A} = (X_e - \mu_a + \frac{A_e}{2})^2 + \frac{1}{3} (\frac{A_e}{2})^2$$

The fourth moment constant will be developed by rewriting equation (A-7) and ignoring the approximation

$$Z_N{}^4 = \frac{N_e}{N} [(X_e - \mu_a + A_e)^4 - (X_e - \mu_a)^4]$$

$$= \frac{N_e}{N} [4A_e(X_e - \mu_a)^3 + 6A_e^2 (X_e - \mu_a)^2 + 4A_e^3 (X_e - \mu_a) + A_e^4]$$

$$= \frac{4N_eA_e}{N} [(X_e - \mu_a)^3 + \frac{3}{2} A_e(X_e - \mu_a)^2 + A_e^2 (X_e - \mu_a) + \frac{A_e^3}{4}]$$

$$= \frac{4N_eA_e}{N} [(X_e - \mu_a)^3 + 3(\frac{A_e}{2}) (X_e - \mu_a)^2 + 3(\frac{A_e}{2})^2 (X_e - \mu_a)$$

$$+ \frac{1}{2} \frac{A_e^3}{4} + \frac{A_e^2}{4} (X_e - \mu_a) + \frac{1}{2} \frac{A_e^3}{4}]$$

$$= \frac{4N_eA_e}{N} [(X_e - \mu_a + \frac{A_e}{2})^3 + (\frac{A_e}{2})^2 (X_e - \mu_a + \frac{A_e}{2})]$$

$$Z_N{}^4 = \frac{4N_eA_e}{N} (X_e - \mu_a + \frac{A_e}{2}) [(X_e - \mu_a + \frac{A_e}{2})^2 + (\frac{A_e}{2})^2] \qquad \text{(B-3)}$$

Rewriting equation (A-8) and substituting equation (B-3) into it yields

$$\mu_{4a} - \mu_{4r} + 4\mu_A \mu_{3r} - 6\mu_A^2 \sigma_r^2 - \mu_A^4 =$$

$$\frac{4N_eA_e}{N} (X_e - \mu_a + \frac{A_e}{2}) [(X_e - \mu_a + \frac{A_e}{2}) + (\frac{A_e}{2})^2]$$

Dividing both sides of the equation by $4N_eA_e/N$, the fourth moment constant is obtained.

$$M_4 = \frac{\mu_{4a} - \mu_{4A} + 4\mu_A \mu_{3r} - 6\mu_A^2 \sigma_r^2 - \mu_A^4}{4 N_eA_e/N} \qquad \text{(B-4)}$$

$$= (X_e - \mu_a + \frac{A_e}{2}) [(X_e - \mu_a + \frac{A_e}{2})^2 + (\frac{A_e}{2})^2]$$

But $N_e A_e / N = \mu_A$ and

$$M_4 = \frac{\mu_{4a} - \mu_{4r} + 4\mu_A\ \mu_{3r} - 6\mu_A{}^2\ \sigma_r{}^2 - \mu_A{}^4}{4\mu_A}$$

$$= (X_e - \mu_a + \frac{A_e}{2})\ [(X_e - \mu_a + \frac{A_e}{2})^2 + (\frac{A_e}{2})^2]$$

Referring to equation (A-9) the $n^{th}$ moment constant becomes

$$M_n = \frac{\mu_{na} - \frac{1}{N}\sum_{1}^{N} (y_i - \mu_A)^n}{n\mu_A} = \frac{Z_N^n}{n\mu_A} \qquad \text{(B-5)}$$